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Mathematical Analysis 2

Midterm 1

1. Solve the differential equation:

$$xy^2 = y'e^{-x} - x \ .$$

SOLUTION: This equation is separable.

$$xe^x y^2 = y' - xe^x \implies y' = xe^x(1+y^2) \implies$$
$$\frac{1}{1+y^2}dy = xe^x dx \implies \int \frac{1}{1+y^2}dy = \int xe^x dx \implies$$
$$\arctan y = (x-1)e^x + C \implies \boxed{y = \tan\big((x-1)e^x + C\big)} \ .$$

2. Solve the initial value problem:

$$xy' = y + \frac{x}{\cos\frac{y}{x}} \ , \qquad y(1) = \frac{\pi}{4} \ .$$

SOLUTION: This equation can be written as $y' = \frac{y}{x} + \frac{1}{\cos\frac{y}{x}}$, so it is homogeneous, and requires the substitution $y = xv$, $y' = xv' + v$. Thus we get

$$xv' + \cancel{v} = \cancel{v} + \frac{1}{\cos v} \implies \cos v dv = \frac{dx}{x} \implies \int \cos v dv = \int \frac{dx}{x} \implies$$
$$\sin v = \ln|x| + C \implies v = \arcsin(\ln|x| + C) \implies \boxed{y = x \arcsin(\ln|x| + C)} \ .$$

Now we plug in $x = 1$ and get

$$y(1) = \frac{\pi}{4} = \arcsin C \implies C = \sin(\tfrac{\pi}{4}) \implies \boxed{C = \tfrac{\sqrt{2}}{2}} \ .$$

Remark: The solution is valid only for x's larger than 0 and such that $\ln x + \frac{\sqrt{2}}{2} \in [-1, 1]$, i.e., for $x \in \left[e^{-1-\frac{\sqrt{2}}{2}}, e^{1-\frac{\sqrt{2}}{2}}\right]$.

3. Solve the initial value problem:

$$y' = \tfrac{5}{4}x - y - (\tfrac{5}{4}x - y)^2 \,, \qquad y(0) = \frac{1}{2} \,.$$

SOLUTION: This equation requires the linear substitution $u = \frac{5}{4}x - y$, which yields $y' = \frac{5}{4} - u'$. Then we obtain:

$$\tfrac{5}{4} - u' = u - u^2 \implies u' = u^2 - u + \tfrac{5}{4} \implies \frac{du}{u^2 - u + \frac{5}{4}} = dx \implies$$

$$\int \frac{du}{u^2 - u + \frac{5}{4}} = \int dx \implies \int \frac{du}{(u - \frac{1}{2})^2 + 1} = x + C \implies$$

$$\arctan(u - \tfrac{1}{2}) = x + C \implies u = \tan(x + C) + \tfrac{1}{2} \implies \boxed{y = \frac{5}{4}x - \tan(x + C) - \frac{1}{2}} \,.$$

Now we plug in $x = 0$ and get

$$y(0) = \tfrac{1}{2} = -\tan(C) - \tfrac{1}{2} \implies \tan(C) = -1 \implies \boxed{C = -\frac{\pi}{4}} \,.$$

4. Solve the differential equation:

$$y' = \frac{y}{x} + \frac{x^2}{x^2 + 2} \,.$$

SOLUTION: This is a nonhomogeneous linear equation. We start by solving the homogeneous equation $y' = \frac{y}{x}$, i.e., $\frac{dy}{y} = \frac{dx}{x}$, which solves to $\ln|y| = \ln|x| + C$. Applying the exponential function we get $y = xC$, where the new C is the old $\pm e^C$. To solve the nonhomogeneous equation we apply variation of the constant, i.e., we replace the constant C by a function $c(x)$. This leads us to $y = xc(x)$ and $y' = xc'(x) + c(x)$ (we will write just $y' = xc' + c$). Plugging this into the original equation, we get

$$xc' + \cancel{c} = \cancel{c} + \frac{x^2}{x^2 + 2} \implies dc = \frac{x^{\cancel{2}}}{\cancel{x}(x^2 + 2)}dx \implies \int dc = \int \frac{x}{x^2 + 2}dx \implies$$

$$c = \frac{1}{2}\int \frac{2x}{x^2 + 2}dx \implies c = \tfrac{1}{2}\ln(x^2 + 2) + C \implies \boxed{y = x\left(\tfrac{1}{2}\ln(x^2 + 2) + C\right)} \,.$$

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